Fall 2019, Stat 426 : Homework 1

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Announcement: The homework carries 65 points and is due on Sep 27. Time and place for turning in HW to be announced later.

1. Consider a population of size N of individuals marked $\{1, 2, \ldots, N\}$ where the i'th individual has a trait that is labeled as 0, 1 or 2 (for example, these could be stages of a medical condition). Let p_0, p_1 and p_2 be the proportions corresponding to these three traits: thus $p_0 = N_0/N$ where N_0 of the N individuals has trait 0, and so on and so forth. Consider a random sample of size n drawn without replacement from this population, and let X_i denote the value of the trait for the i'th individual in the sample. Let $n_l = \sum_{i=1}^{n} 1(X_i = l)$ for $l = 0, 1, 2$. Thus n_0 is the number of individuals in the sample with trait 0 and so on and so forth. Write $\hat{p}_l = n_l/n$.

   (a) Argue that $\hat{p}_l$ is a reasonable estimate of p_l. Calcuate the mean and variance of p_l.
   (b) Find an explicit formula for $P(X_{i_1} = \epsilon_1, X_{i_2} = \epsilon_2, \ldots, X_{i_k} = \epsilon_k)$ for some $k \leq n$ and $i_1 < i_2 < \ldots < i_k$, in terms of $p_0, p_1, p_2, N, s_0, s_1, s_2$ where s_l is the number of ϵ_j's that equal l.
   (c) Find the covariance between $\hat{p}_0$ and $\hat{p}_1$. How does it compare to the covariance between $\hat{p}_0$ and $\hat{p}_1$ in the case that $X_1, X_2, \ldots, X_n$ are sampled WITH replacement? [7 + 8 + 10 = 25 points]

2. A geometric random variable W takes values $\{1,\ 2,\ 3,\ .\ .\ .\}$ and $P(W = j) = \theta(1-\theta)^{j-1}$,where $0 < \theta < 1$.

   (a) Prove that for any two positive integers i, j, it is the case that, $P(W > i + j | W > i) = P(W > j)$.

   (b) Indeed, the converse is also true. We show that if W is a discrete random variable taking values $\{1, 2, 3, .\ .\ .\}$ with probabilities $\{p_1, p_2, p_3, .\ .\ .\}$ and satisfies the memoryless property, then W must follow a geometric distribution.

      Follow these steps to establish the fact that W is geometric. Using the fact that W has the memoryless property, show that

      $$P(W > m) = (P(W > 1))^m,$$

      for any $m \geq 2$. As a first step towards proving this show that

      $$P(W > 2) = (P(W > 1))^2$$

Define $\theta = P(W = 1)$ and $1 - \theta = P(X > 1)$. You now have,

$$P(W > m) = (1 - \theta)^m,$$

for any $m \geq 2$. Use this to show that for any $m \geq 2$,

$$P(W = m) = \theta(1 - \theta)^{m-1}.$$

But for $m = 1$, certainly

$$P(W = m) = P(W = 1) = \theta = \theta(1 - \theta)^{m-1},$$

and the proof is complete. (10 points)

3. Let T be an exponential random variable with parameter β and let W be a random variable independent of T which assumes the value 1 with probability 2/3 and the value -1 with probability 1/3. Find the density of $X = WT$.

   Hint: It would help to split up the event $\{X \leq x\}$ as the union of $\{X \leq x, W = 1\}$ and $\{X \leq x, W = -1\}$. (10 points)

4. Let X be an Exp(1) random variable. Let $[X]$ denote the largest integer not exceeding X. Show that

$$P([X] = m, X - [X] \leq t) = e^{-\lambda m}(1 - e^{-\lambda t}).$$

   Work out the marginals of $[X]$ and $X - [X]$ and deduce that these two random variables are independent. (10 points)

5. (a) (a) If X_1 and X_2 are independent Poisson random variables with parameters λ_1 and λ_2, then show that $X_1 + X_2$ is also Poisson with parameter $\lambda_1 + \lambda_2$. Recall that if W follows Poisson(θ), then the p.m.f. of W is,

$$P(W = m) = \frac{e^{-\theta}\theta^m}{m!}$$

   Hint: Write $P(X_1 + X_2 = m)$ as $\sum_{i=0}^{m} P(X_1 = i, X_2 = m - i)$ and proceed. (5 points)

   (b) Use this result repeatedly to show that if $X_1, X_2, ..., X_n$ are independent Poisson random variables with parameters $\lambda_1, \lambda_2, ..., \lambda_n$ respectively, then $X_1 + X_2 + ... + X_n$ follows Poisson with parameter $\lambda_1 + \lambda_2 + ... + \lambda_n$. (5 points)

(c) Show from first principles that the conditional distribution of the random vector $(X_1, X_2, ..., X_n)$ given that the sum $X_1 + X_2 + ... + X_n = K$ for some integer K follows the multinomial distribution with parameters $(K, p_1, p_2, ..., p_n)$ where each p_i is given by $\frac{\lambda_i}{\sum_{j=1}^{m} \lambda_j}$. (10 points)